Journal of Pure and Applied Algebra 50 (1988) 73–79
North-Holland

GABRIEL DIMENSION OF GRADED RINGS, II

Constantin NĂSTĂSESCU and Şerban RAIANU
Facultatea de Matematică, R 70109 Bucharest 1, Romania

Communicated by C.A. Weibel
Received 10 April 1985

Introduction

Although this note is a sequel of our paper [7], the technique we use now is completely different, being very much inspired by the one we have used in the proof of [8, Theorem 2.2] (see the key result, Proposition 2.1).

In [7] we have studied the following problem: does a graded module with graded Gabriel dimension have a Gabriel dimension when regarded without grading? We gave in [7] a positive answer to this problem in the commutative case, as well as a relation between the two dimensions, leaving the problem open in the non-commutative case. The purpose of this note is to settle the problem completely, and to use the same (adapted) simple argument in order to obtain an evaluation for the Gabriel dimension of polynomials, previously proved by Gordon and Robson [5] using polynomial categories. The reader may note that our proof avoids the use of quotient categories and polynomial categories.

1. Notation and preliminaries

All rings considered in this paper will be unitary. If R is a ring, by an R-module we will mean a left R-module, and we will let R-**mod** denote the category of R-modules. When R will be supposed to be graded, this will mean that R is a graded ring of type $\mathbb{Z}$, and R-**gr** will denote the category of all graded R-modules. If $M \in R$-**gr**, we put

$$h(M) = \{x \in M \mid x \text{ is homogeneous}\}.$$

Let R be a ring and let F be a (left) Gabriel topology on R [12]. For each $M \in R$-**mod**, we denote by $C_F(M)$ the modular lattice

$$C_F(M) = \{N \le M \mid M/N \text{ is } F\text{-torsion free}\} \text{ [2]}.$$

M is said to be *F-noetherian* if $C_F(M)$ is a noetherian lattice.

The graded analogue of a Gabriel topology is the notion of a graded filter [10]. If $M \in R$-**gr** and H is a graded filter on R, we put

$$C_H^g(M) = \{N \leq M \mid N \text{ is graded and } M/N \text{ is gr-}H\text{-torsion free}\} .$$

Then M is said to be gr-*H-noetherian* if $C_H^g(M)$ is a noetherian lattice [8].

We recall now the notions of relative Krull dimension and Gabriel dimension. The accord with other definitions of the Gabriel dimension may be found in [1, 6].

Let $(L, \leq)$ be a partially ordered set. We put

$$\Gamma(L) = \{(a, b) \mid a \leq b,\, a, b \in L\} .$$

We define by transfinite recursion the following filtration:

$$\Gamma_{-1}(L) = \{(a, b) \in \Gamma(L) \mid a = b\} ,$$
$$\Gamma_0(L) = \{(a, b) \in \Gamma(L) \mid [a, b] \text{ is artinian}\} .$$

Suppose that $\Gamma_\beta(L)$ is defined for all ordinals $\beta < \alpha$, and let $\Gamma_\alpha(L) = \{(a, b) \in \Gamma(L) \mid \forall b \geq b_1 \geq \cdots \geq b_k \geq \cdots \geq a, \exists n \in \mathbb{N}$ such that $(b_{i+1}, b_i) \in \bigcup_{\beta<\alpha} \Gamma_\beta(L)$, $\forall i \geq n\}$. If there exists an ordinal α such that $\Gamma(L) = \Gamma_\alpha(L)$, then we say that the Krull dimension of L is defined. If so, then the least such ordinal is called *the Krull dimension of* L, and will be denoted by K.dim(L).

Let R be a (graded) ring, $M \in R$-**mod** (resp. R-**gr**) and F (resp. H) a Gabriel topology (resp. a graded filter) on R. Then *the (graded) Krull dimension of* M *with respect to* F *(resp.* H*)* is defined as $\text{K.dim}_F(M) = \text{K.dim}(C_F(M))$ (resp. gr-$\text{K.dim}_H(M) = \text{K.dim}(C_H^g(M))$). Taking $F = \{R\}$ (resp. $H = \{R\}$), we obtain the well-known notions of Krull dimension (resp. graded Krull dimension).

We will need the following:

Lemma 1.1 (Gabriel and Rentschler [4]). *Let* $(E, \leq)$ *be an ordered set having a least and a greatest element. Denote by* $S_c(E)$ *the set of stationary chains of elements of* E, *and by* $A_c(E)$ *the subset of* $S_c(E)$ *consisting of all ascending chains. We order* $S_c(E)$ *by putting* $(e_i)_{i\in\mathbb{N}} \leq (f_i)_{i\in\mathbb{N}} \Leftrightarrow e_i \leq f_i$, $\forall i \in \mathbb{N}$. *If* E *has a Krull dimension, then so do* $S_c(E)$ *and* $A_c(E)$. *Moreover,*

$$\text{K.dim}(S_c(E)) = \text{K.dim}(A_c(E)) = \text{K.dim}(E) + 1 . \qquad \square$$

Let now α be an ordinal and L a lattice with '0' and '1'. L is said to be α-*critical* if K.dim$(L) = \alpha$ and K.dim$([a, 1]) < \alpha$ for each $a \in L$, $a \neq 0$. An element $a \in L$ is said to be α-critical if $[0, a]$ is α-critical. It is well known that a lattice with Krull dimension has a critical element.

Let now L be a modular lattice, upper continuous, with '0' and '1'. We define *the Gabriel dimension of* L by transfinite recursion: $\mathrm{G.dim}(L)=0$ if and only if $L=\{0\}$. Suppose that α is not a limit ordinal and that $\mathrm{G.dim}(L')$ has been defined for each L' with $\mathrm{G.dim}(L')<\alpha$. We say that L is *α-simple* if for each $a\in L$, $a\neq 0$, we have $\mathrm{G.dim}([0,a])\not<\alpha$ and $\mathrm{G.dim}([a,1])<\alpha$. We put $\mathrm{G.dim}(L)=\alpha$ if $\mathrm{G.dim}(L)\not<\alpha$, and for each $a\neq 1$ there exists $b\neq a$ such that $[a,b]$ is β-simple for some $\beta\leq\alpha$.

If α is a limit ordinal, then $\mathrm{G.dim}(L)=\alpha$ if $\mathrm{G.dim}(L)\not<\alpha$ and for each $a\neq 1$ there exists $b\neq a$ such that $[a,b]$ is β-simple for some $\beta<\alpha$.

From the definition it follows that if L is α-simple, then $\mathrm{G.dim}(L)=\alpha$ and α is not a limit ordinal, and that $\mathrm{G.dim}(L)\leq\alpha$ if and only if for each $a\in L$, $a\neq 1$, $[a,1]$ contains a β-simple element ($b\in L$ is said to be β-simple if $[0,b]$ is β-simple) for some $\beta\leq\alpha$.

If R is a (graded) ring and $M\in R$-**mod** (resp. R-**gr**), then the *(graded) Gabriel dimension of* M, $\mathrm{G.dim}(M)$ (resp. gr-$\mathrm{G.dim}(M)$) is equal to the Gabriel dimension of the lattice of all (graded) submodules of M [1].

It is well known that the (graded) R-modules having (graded) Gabriel dimension $\leq\alpha$ form a localizing subcategory of R-**mod** (resp. R-**gr**) for each ordinal α. We denote the corresponding Gabriel topology (resp. graded filter) by F_α (resp. H_α). In particular, $F_0=H_0=\{R\}$. Note that if M is α-simple (resp. gr-α-simple), then $\mathrm{K.dim}_{F_{\alpha-1}}(M)=0$ (resp. gr-$\mathrm{K.dim}_{H_{\alpha-1}}(M)=0$.

Our next aim is to prove a relation between the relative Krull dimension and the Gabriel dimension. We remark that this result, essentially due to Gabriel [3], has an obvious categorical proof. Nevertheless, we prefered to provide a non-categorical one. First we need

Lemma 1.2. *Let R be a graded ring and M a (graded) R-module such that M/N has a (graded) Gabriel dimension for each (graded) R-submodule N of M, $N\neq 0$. Then* (gr-)$\mathrm{G.dim}(M)\leq\alpha+1$, *where*

$$\alpha=\sup\{(\text{gr-})\mathrm{G.dim}(M/N)\mid 0\neq N\leq M\ (N\text{ is graded})\}\,. \qquad \square$$

Proposition 1.3 (Gabriel [3]). *Let R be a (graded) ring and F (resp. H) a Gabriel topology (resp. a graded filter) on R such that each F (resp. H) -torsion (graded) R-module X has* (gr-)$\mathrm{G.dim}(X)\leq\alpha$. *If M is an F-noetherian (resp.* gr-*H-noetherian) (graded) R-module having* $\mathrm{K.dim}_F(M)\leq\beta$, *(resp.* gr-$\mathrm{K.dim}_H(M)\leq\beta$), *then M has a (graded) Gabriel dimension, and*

$$(\text{gr-})\mathrm{G.dim}(M)\leq\alpha+\beta+1\,.$$

Proof. Since the proof of the graded case is identical to the one of the ungraded case, we only prove the latter. We do this by transfinite recursion on β. It is obvious that we may assume that M is F-torsion free.

If $\beta = 0$, then $\mathrm{K.dim}_F(M) = 0$, and hence M has a series $0 = N_0 < N_1 < \cdots < N_k = M$, whose factors are F-simple. It is therefore sufficient to consider the case that M is F-simple, i.e. $C_F(M)$ is 1-simple. Thus $\mathrm{G.dim}(M/N) \le \alpha$, $\forall 0 \ne N \le M$, since M/N is F-torsion, and hence $\mathrm{G.dim}(M) \le \alpha + 1$ by Lemma 1.2.

We suppose now that the assertion is true for all ordinals $< \beta$, and prove it for β.

Let $0 \ne X$ be a factor of M. We will prove that X contains a γ-simple R-module, $\gamma \le \alpha + \beta + 1$. The case when X is not F-torsion free being clear, we assume that $X = M/N$, $N \in C_F(M)$. It follows that $\mathrm{K.dim}_F(X) \le \beta$ and hence there exists $P \in C_F(X)$ such that P is a θ-critical element of the lattice $C_F(X)$, $\theta \le \beta$. We prove that $\mathrm{G.dim}(P) \le \alpha + \beta + 1$.

Let $0 \ne Q \le P$. If $Q \in C_F(P)$, then $\mathrm{K.dim}_F(P/Q) = \varepsilon < \theta \le \beta$, and by the induction hypothesis we have $\mathrm{G.dim}(P/Q) \le \alpha + \varepsilon + 1 \le \alpha + \beta$. If $Q \not\in C_F(P)$, then $t(P/Q) = T/Q \ne 0$, and $T \in C_F(P)$ (t is the torsion radical of F [12]). From the exact sequence

$$0 \to t(P/Q) \to P/Q \to P/T \to 0$$

we deduce that $\mathrm{G.dim}(P/Q) \le \alpha + \beta$. Hence $\mathrm{G.dim}(P) \le \alpha + \beta + 1$ by Lemma 1.2 and the proof is complete. □

We end this section by recalling some elementary facts about External Homogenization, as may be found in [10].

Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded ring. The ring of poynomials $R[X]$ is a graded ring if we put $\deg(X) = 1$ and for each $k \in \mathbb{Z}$

$$R[X]_k = \left\{ \sum_{i+j=k} a_i X^j \mid a_i \in R_i \right\}.$$

The same construction may be performed on $M[X] = R[X] \otimes_R M$ starting from $M \in R$-**gr**.

Write $x \in M$ as a sum of its homogeneous components, say

$$x = x_{-m} + \cdots + x_0 + \cdots + x_n ,$$

and let

$$x^* = x_{-m}X^{m+n} + \cdots + x_0 X^n + \cdots + x_n \in M[X]_n .$$

If N is a submodule of M, we take N^* to be the graded $R[X]$-submodule of $M[X]$ generated by all x^*, $x \in N$. It is well known that if $P \subsetneq N$, then $P^* \subsetneq N^*$.

Grading $R[X]$ as above, the inclusion $i : R \to R[X]$ becomes a graded morphism of degree zero. If H (resp. F) is a graded filter (resp. a Gabriel topology) on R, we will let $i(H)$ (resp. $i(F)$) denote the direct image of H (resp. F) by i, which is a graded filter (resp. a Gabriel topology) on $R[X]$:

$i(H)=\{J$ graded left ideal of $R[X] \mid (J:y)\cap R\in H,\ \forall y\in h(R[X])\}$ [8] (resp. $i(F)=\{I\leq R[X] \mid (I:x)\cap R\in F,\ \forall x\in R[X]\}$ [2]).

All other unexplained facts and notation concerning Graded Ring Theory (resp. Torsion Theory) may be found in [10] (resp. [12]). The reader is referred to [5, 6] for details on the Gabriel dimension.

2. Gabriel dimension of graded modules

In this section we prove that a graded module having a graded Gabriel dimension must have a Gabriel dimension when regarded without grading (Theorem 2.2). The key result for proving this theorem is the following:

Proposition 2.1. *Let R be a graded ring, H a graded filter on R, and M a graded R-module which is* gr-H*-noetherian and has* $\text{gr-K.dim}_H(M)=\alpha$. *Let F be the smallest Gabriel topology containing H. Then* $\text{K.dim}_F(M)$ *exists, and we have*

$$\alpha\leq K.dim_F(M)\leq \alpha+1\ .$$

Proof. We will show the existence of $\text{K.dim}_F(M)$ in two steps:

Step 1. We show first the existence of $\text{gr-K.dim}_{i(H)}(M[X])$. This is essentially based on [8, Proposition 2.1], which asserts that $M[X]$ is gr-$i(H)$-noetherian. We construct a strictly increasing mapping from $C^{g}_{i(H)}(M[X])$ to $A_{c}(C^{g}_{H}(M))$ by associating to each $N\in C^{g}_{i(H)}(M[X])$ the stationary ascending sequence $(N_0^{\sim}, N_1^{\sim},\dots,N_k^{\sim},\dots)$, where $N_k=\{x\in M \mid \exists xX^k+\cdots+x_k\in N\}$, and $N_k^{\sim}=\{x\in M \mid (N_k:x)\in H\}$. Then it may be shown like in the proof of [1, 1.12] that if $N\subsetneq P\in C^{g}_{i(H)}(M[X])$, then $N_k^{\sim}\subsetneq P_k^{\sim}$ for all $k\in\mathbb{N}$, and $(N_0^{\sim}, N_1^{\sim},\dots)\neq(P_0^{\sim},P_1^{\sim},\dots)$. Thus $\text{gr-K.dim}_{i(H)}(M[X])=\text{K.dim}(C^{g}_{i(H)}(M[X]))\leq \text{K.dim}(A_{c}(C^{g}_{H}))=\text{K.dim}(C^{g}_{H}(M))+1=\alpha+1$ by Lemma 1.1. Thus $\text{gr-K.dim}_{i(H)}(M[X])$ exists, and $\text{gr-K.dim}_{i(H)}(M[X])\leq\alpha+1$.

Step 2. We show now that $\text{K.dim}_F(M)\leq\text{gr-K.dim}_{i(H)}(M[X])$. This was essentially done in the proof of [8, Theorem 2.2], where it is proved that if $N\in C_F(M)$, then $N^*\in C^{g}_{i(H)}(M[X])$. Since the mapping $N\mapsto N^*$ is strictly increasing, it follows that $\text{K.dim}_F(M)\leq\text{gr-K.dim}_{i(H)}(M[X])\leq\alpha+1$.

In order to prove the left inequality in the statement, we remark that $C^{g}_{H}(M)\subseteq C_F(M)$. A proof of this simple result may be found in [11]. □

We are now in a position to state and prove the main result of this note,

Theorem 2.2. *Let R be a graded ring and $M\in R$-**gr** having* $\text{gr-G.dim}(M)=\varepsilon+n$, *where $\varepsilon=0$, or ε is a limit ordinal, and $n\in\mathbb{N}$. Then* $\text{G.dim}(M)$ *exists, and we have*

$$\varepsilon+n\leq \text{G.dim}(M)\leq\varepsilon+2n\ .$$

Proof. The left inequality being obvious, we show the existence of G.dim(M). We will prove that $H_{\varepsilon+n} \subseteq F_{\varepsilon+2n}$ by transfinite recursion on $\alpha = \varepsilon + n$.

If $\alpha = 1$, then gr-G.dim$(M) = 1$, and we reduce ourselves to the case M gr-simple. Then gr-K.dim$(M) = 0$, and hence K.dim$(M) \leq 1$ by Proposition 2.1. Thus G.dim$(M) \leq 2$ by Proposition 1.3.

We suppose now that the assertion is true for all ordinals $< \alpha = \varepsilon + n$, and prove it for α. If $\alpha = \varepsilon + n$, $n \neq 0$, then we reduce the problem to the case that M is gr-α-simple, hence we may assume gr-K.dim$_{H_{\varepsilon+n-1}}(M) = 0$. Hence K.dim$_{F_{\varepsilon+2n-2}}(M) \leq 1$ by Proposition 2.1 and the induction hypothesis. Thus G.dim$(M) \leq \varepsilon + 2n - 2 + 1 + 1 = \varepsilon + 2n$ by Proposition 1.3.

Now if $\alpha = \varepsilon$ is a limit ordinal, then $M = \bigcup_{\lambda < \varepsilon} M_\lambda$, where $M_\lambda \in R$-**gr** and gr-G.dim$(M_\lambda) \leq \lambda$ for each $\lambda < \varepsilon$. It follows by the induction hypothesis that G.dim$(M_\lambda) < \varepsilon$, $\forall \lambda < \varepsilon$, and thus G.dim$(M) \leq \alpha$. □

Remark 2.3. Theorem 2.2 also holds for graded modules over rings graded by a finitely generated abelian group, the proof remaining the same. The only difference is that one should use an adapted External Homogenization which may be found in [9] by the interested reader.

3. Polynomials

The purpose of this section is to provide a new simple proof of the following result, due to Gordon and Robson:

Theorem 3.1. *Let M be an R-module having* G.dim$(M) = \varepsilon + n$, *where $\varepsilon = 0$ or ε is a limit ordinal, and $n \in \mathbb{N}$. Then*

$$\begin{aligned} \varepsilon + n + 1 \leq \mathrm{G.dim}(M[X]) &\leq \varepsilon + 2n \quad && \text{if } n \neq 0, \\ \mathrm{G.dim}(M[X]) &= \varepsilon && \text{if } n = 0. \end{aligned}$$

The proof of the existence of G.dim$(M[X])$ will be identical to the one of the existence of G.dim(M) in Theorem 2.2, but first we need an analogue of Proposition 2.1:

Proposition 3.2. *Let R be a ring, F a Gabriel topology on R, and $M \in R$-**mod** such that M is F-noetherian and* K.dim$_F(M) = \alpha$. *Then*

$$\mathrm{K.dim}_{i(F)}(M[X]) = \alpha + 1 .$$

Proof. We will show that K.dim$_{i(F)}(M[X]) = $ K.dim$(A_c(C_F(M))) = \alpha + 1$ (see Lemma 1.1).

Let first $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k \subseteq \cdots$ be a stationary chain, where $N_k \in C_F(M)$ for all $k \in \mathbb{N}$. Then $N = N_0 \oplus XN_1 \oplus \cdots \oplus X^k N_k \oplus \cdots \in C_{i(F)}(M[X])$. Indeed, we suppose that this is false and look for a contradiction. There exists $x \not\subseteq N$ such that $(N:x)_{R[X]} \in i(F)$. It follows that $(N:x)_{R[X]} \cap R \in F$. Let $x = x_0 + Xx_1 + \cdots + X^k x_k$, and let $0 \leq i \leq k$ such that $x_i \not\subseteq N_i$. Then $(N:x)_{R[X]} \cap R \subseteq (N_i : x_i)_R$, and hence $(N_i : x_i)_R \in F$, a contradiction. It follows that $\alpha + 1 = \text{K.dim}(A_c(C_F(M))) \leq \text{K.dim}_{i(F)}(M[X])$.

The proof of the other inequality is nothing but a mere transcription of the proof of Step 1 of Proposition 2.1. $\square$

Proof of Theorem 3.1. Like the proof of Theorem 2.2, using Proposition 3.2 instead of Proposition 2.1. Note that the inequality $\varepsilon + n + 1 \leq \text{G.dim}(M[X])$ for $n \neq 0$ follows directly from Proposition 3.2 taking $F = \{R\}$, since the inequality $\text{K.dim}_{i(F)}(M[X]) \geq \text{K.dim}_F(M) + 1$ does not require chain conditions on M. $\square$

References

[1] T. Albu, Gabriel dimension of partially ordered sets, I, II, Bull. Soc. Sci. Math. R.S.R. 28 (76), (2) (1984) 100–103, (3) (1984) 200–205.

[2] T. Albu and C. Năstăsescu, Décompositions primaires dans les catégories de Grothendieck commutatives, I, II, J. Reine Angew. Math. 280, 282 (1976) 172–194, 172–185.

[3] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.

[4] P. Gabriel and R. Rentschler, Sur la dimension des anneaux et ensembles ordonées, C.R. Acad. Sci. Paris Sér. A 265 (1967) 712–715.

[5] R. Gordon and J.C. Robson, The Gabriel dimension of a module, J. Algebra 29 (1974) 459–473.

[6] C. Năstăsescu, Teoria dimensiunii în algebra necomutativă, Editura Academiei R.S.R., Bucharest, 1983.

[7] C. Năstăsescu and Ş. Raianu, Gabriel dimension of graded rings I, Rend. Sem. Mat. Univ. Padova 71 (1984) 195–208.

[8] C. Năstăsescu and Ş. Raianu, Finiteness conditions for graded modules (gr-$\Sigma(\Delta)$-injective modules), J. Algebra, 100 (1) (1986) 179–190

[9] C. Năstăsescu and Ş. Raianu, External homogenization for finitely generated abelian groups, Analele Univ. Jassy 31 S.Ia (1985) 58–60.

[10] C. Năstăsescu and F. Van Oystaeyen, Graded Ring Theory (North-Holland, Amsterdam, 1982).

[11] Ş. Raianu, A remark on strongly graded rings with finiteness conditions, Bull. Soc. Sci. Math. R.S.R. 30 (78) (3) (1986) 265–269.

[12] B. Stenström, Rings of Quotients (Springer, Berlin, 1975).